a general dispersion relation. Worthwhile to investigate it as it was, we felt it however not feasable without additional physical approximations or impossibly difficult mathematical recursion procedures to give more explicit formulas in the most general case.

For the principal waves however, propagating along or across the external magnetic induction, some short formulas were found. A discussion of these results compared with existing work in the field showed the inclusion of many known formulas as special cases of our relations. The present form of these relations makes them easily suitable for various other physical approximations, which can be as varied as the existing literature on plasma waves. This, however, is beyond the scope of this article.

At the end of this paper, Professor dr. R. MERTENS and my colleague A. BROUCKE should be thanked for their interest and helpful discussions.

Multicomponent Beam-Plasma Waves

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(Z. Naturforschg. 22 a, 1935—1939 [1967]; received 14 July 1967)

The linearization procedure is applied to the equations governing a beam-plasma system, in which the stream velocities and the wavevector are parallel to the external magnetic induction. No special constraints are imposed on the parameters characterizing the constituent fluids in the equilibrium state of this macroscopic picture. From the MAXWELL equations an expression for the electromagnetic field of the wave is obtained and substituted in the equations of motion. The components of the first-order pressure tensors are computed in the low-temperature approximation, but without recurring to the strong magnetic induction CGL hypothesis. Since the equations of motion are now expressed only in the components of the perturbations of the drift velocities, the dispersion relations follow immediately. These relations are applicable to all beam-plasma systems comprised between the now conventional multicomponent plasma and the system of beams of charged particles. Some known cold beam-plasma cases are included in the general dispersion equations.

1. Introduction

In our investigation of multicomponent beam-plasma waves, we will adhere as closely as possible to the treatment of waves in multicomponent VLASOV plasmas without zero-order drift velocities[1] (henceforth referred to as I). We will in particular make use of the same notations and formulas of I if possible, and refer to them as to (I ...).

The set of basic equations (I.2.6—8) will be linearized here in a similar fashion as in I, with the help however of the following additional hypotheses.

(1) In our macroscopic picture of the plasma a zero-order drift velocity is introduced for every constituent fluid. We now must designate by „component of the beam-plasma system" the collection of all particles which have the same values for the complete set of characterizing zero-order parameters, such as charge, mass, density, equilibrium drift velocity and equilibrium pressure. Electrons with different beam velocities hence belong to different components of the system. Furthermore, at any stage of the analysis some of these finite drift velocities can be put equal to zero to describe the pure plasma part of the system. The results thus will be applicable to streaming multicomponent plasmas, beams of charged particles and every other combination of these plasmas and beams.

(2) The most striking features of the introduction of finite drift velocities are noted in the direction of the external magnetic induction, if present, and of the wavevector. We therefore restrict ourselves in this study to wave propagation parallel to the external magnetic induction. For mathematical simplicity we direct the finite drift velocities along the now privileged z-axis:

$$\boldsymbol{V}_a = V_a \hat{\boldsymbol{z}}, \quad \boldsymbol{B} = B \hat{\boldsymbol{z}}, \quad \boldsymbol{k} = k \hat{\boldsymbol{z}} \qquad (1.1)$$

(3) A scalar equilibrium pressure is adopted to avoid too intricate formulas, changing (I.3.2) into

$$\mathbf{P}_a = P_a \, \mathbf{I} \qquad (1.2)$$

for every $a = 1, \ldots, N$. Our treatment, however, still caters for anisotropic pressure variations. As we do

[1] F. G. VERHEEST, Z. Naturforschg. 22a, 1927 [1967].

not use the CHEW-GOLDBERGER-LOW approximation[2] for the pressure tensors, our results are valid for unmagnetized beam-plasma systems as well as for magnetized ones.

(4) The system in the equilibrium state must, taken as a whole, be electrically neutral and at rest. These requirements, which are necessary for the linearization procedure, are expressed as

$$\sum_a N_a q_a = 0\,, \qquad \sum_a N_a q_a \boldsymbol{V}_a = 0\,.$$

2. Linearization

The linearized forms of the basic equations (I.2.6—8) are written, keeping in mind the supplementary conditions (1.1) and (1.2):

equations of continuity:

$$\omega\, n_a = k (N_a v_3^a + n_a V_a)\,, \tag{2.1}$$

equations of motion:

$$\omega\, \boldsymbol{v}_a - k V_a \boldsymbol{v}_a - \frac{k}{\varrho_a} \sum_{j=1}^{3} p_{3j}^a \hat{\boldsymbol{u}}_j - i \frac{q_a}{m_a} (\boldsymbol{e} + B \boldsymbol{v}_a \times \hat{\boldsymbol{z}} + V_a \hat{\boldsymbol{z}} \times \boldsymbol{b}) = 0\,, \tag{2.2}$$

pressure tensor equations:

$$\omega\, \mathbf{p}_a - k v_3^a P_a \mathbf{I} - k V_a \mathbf{p}_a - k P_a \sum_{j=1}^{3} \sum_{l=1}^{3} (\delta_{3j} v_l^a + \delta_{3l} v_j^a) \hat{\boldsymbol{u}}_j \hat{\boldsymbol{u}}_l + i \Omega_a \sum_{j=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} (\varepsilon_{j3m} p_{ml}^a + \varepsilon_{l3m} p_{mj}^a) \hat{\boldsymbol{u}}_j \hat{\boldsymbol{u}}_l = 0\,. \tag{2.3}$$

Most of the notations used here are conventional and are defined explicitly in I. Newly introduced here are $\hat{\boldsymbol{u}}_j$ and $\hat{\boldsymbol{u}}_l$ to represent respectively unit vectors along the jth and lth axes ($j,\, l = 1, 2, 3$ with $\hat{\boldsymbol{u}}_3 = \hat{\boldsymbol{z}}$); δ_{jl} is the KRONECKER delta and ε_{jlm} stands for the complete antisymmetric LEVI-CIVITÀ unit tensor symbol.

(2.1) can be rewritten as

$$n_a = (k N_a / \omega \varphi_a)\, v_3^a \tag{2.4}$$

if

$$\varphi_a = 1 - k V_a / \omega\,. \tag{2.5}$$

3. Electromagnetic Field

From the MAXWELL equations we obtained in (I.4.2) an expression for the electric field of the small amplitude plane waves

$$\boldsymbol{e} = i \frac{k^2 j_3}{\varepsilon_0 \alpha \omega} \hat{\boldsymbol{z}} - i \frac{\omega \mu_0}{\alpha} \boldsymbol{j} \tag{3.1}$$

where α is given by (I.4.1). In the present analysis $\boldsymbol{j}$ must be replaced by

$$\boldsymbol{j} = \sum_b q_b (N_b \boldsymbol{v}_b + n_b V_b \hat{\boldsymbol{z}})$$

which is equivalent with

$$j_{1,2} = \sum_b \sigma_b v_{1,2}^b\,, \qquad j_3 = \sum_b \frac{\sigma_b}{\varphi_b} v_3^b$$

due to (2.4) and (I.4.4). With the help of (I.4.3), (3.1) yields explicitly

$$\frac{q_a}{m_a} e_{1,2} = - \frac{i \omega}{\omega^2 - c^2 k^2} \frac{\Pi_a^2}{\sigma_a} \sum_b \sigma_b v_{1,2}^b\,, \tag{3.2}$$

$$\frac{q_a}{m_a} e_3 = - \frac{i}{\omega} \frac{\Pi_a^2}{\sigma_a} \sum_b \frac{\sigma_b}{\varphi_b} v_3^b \tag{3.3}$$

In an analogous way we learn from (I.4.6) that the magnetic induction associated with the waves is

$$\boldsymbol{b} = - \frac{i \mu_0 k}{\alpha} \hat{\boldsymbol{z}} \times \boldsymbol{j}\,.$$

Hence we find that

$$\frac{q_a}{m_a} b_1 = \frac{i k}{\omega^2 - c^2 k^2} \frac{\Pi_a^2}{\sigma_a} \sum_b \sigma_b v_2^b, \tag{3.4}$$

$$\frac{q_a}{m_a} b_2 = - \frac{i k}{\omega^2 - c^2 k^2} \frac{\Pi_a^2}{\sigma_a} \sum_b \sigma_b v_1^b, \tag{3.5}$$

$$\frac{q_a}{m_a} b_3 = 0\,. \tag{3.6}$$

The expressions (3.2—5) will be substituted in the equations of motion, once the components of the first-order pressure tensors are found. The computation of these components will be done in the following section.

4. Pressure Tensors

The sets of scalar equations equivalent with the tensorial equation (2.3) are

$$\begin{aligned}
\omega \varphi_a p_{11}^a - k P_a v_3^a \quad - 2 i \Omega_a p_{12}^a &= 0\,, \\
\omega \varphi_a p_{22}^a - k P_a v_3^a \quad + 2 i \Omega_a p_{12}^a &= 0\,, \\
\omega \varphi_a p_{33}^a - 3 k P_a v_3^a &= 0\,, \\
\omega \varphi_a p_{12}^a \quad + i \Omega_a (p_{11}^a - p_{22}^a) &= 0\,, \\
\omega \varphi_a p_{13}^a - k P_a v_1^a \quad - i \Omega_a p_{23}^a &= 0\,, \\
\omega \varphi_a p_{23}^a - k P_a v_2^a \quad + i \Omega_a p_{13}^a &= 0\,.
\end{aligned}$$

[2] G. F. CHEW, M. L. GOLDBERGER, and F. E. LOW, Proc. Roy. Soc. A **236**, 112 [1956].

Solving these sets for the components of the first-order pressure tensors yields

$$p_{11}^a = p_{22}^a = \frac{k\,P_a}{\omega\,\varphi_a}\,v_3^a\,, \tag{4.1}$$

$$p_{33}^a = 3\,\frac{k\,P_a}{\omega\,\varphi_a}\,v_3^a\,, \tag{4.2}$$

$$p_{12}^a = p_{21}^a = 0\,, \tag{4.3}$$

$$p_{13}^a = p_{31}^a = \frac{k\,P_a}{\omega^2\,\varphi_a^2 - \Omega_a^2}\,(\omega\,\varphi_a\,v_1^a + i\,\Omega_a\,v_2^a)\,, \tag{4.4}$$

$$p_{23}^a = p_{32}^a = \frac{k\,P_a}{\omega^2\,\varphi_a^2 - \Omega_a^2}\,(\omega\,\varphi_a\,v_2^a - i\,\Omega_a\,v_1^a)\,. \tag{4.5}$$

5. Equations of Motion

The set of vectorial equations (2.2) is replaced by the equivalent set of scalar equations

$$\omega^2\,\varphi_a\,v_1^a - \frac{k\,\omega}{\varrho_a}\,p_{31}^a - i\,\omega\,\frac{q_a}{m_a}\,e_1 - i\,\omega\,\Omega_a\,v_2^a + i\,\omega\,V_a\,\frac{q_a}{m_a}\,b_2 = 0\,,$$

$$\omega^2\,\varphi_a\,v_2^a - \frac{k\,\omega}{\varrho_a}\,p_{32}^a - i\,\omega\,\frac{q_a}{m_a}\,e_2 + i\,\omega\,\Omega_a\,v_1^a - i\,\omega\,V_a\,\frac{q_a}{m_a}\,b_1 = 0\,,$$

$$\omega^2\,\varphi_a\,v_3^a - \frac{k\,\omega}{\varrho_a}\,p_{33}^a \qquad - i\,\omega\,\frac{q_a}{m_a}\,e_3 = 0\,.$$

The substitution of (3.2—5) and of (4.1—5) into these equations transformes them into

$$\begin{cases} \sum_b (A_{ab}\,v_1^b - i\,F_{ab}\,v_2^b) = 0\,, & (5.1)\\ \sum_b (i\,F_{ab}\,v_1^b + A_{ab}\,v_2^b) = 0\,, & (5.2)\\ \sum_b C_{ab}\,v_3^b = 0 & (5.3) \end{cases}$$

if

$$A_{ab} = \omega^2\,\varphi_a\,\sigma_a\,\delta_{ab}\left(1 - \frac{c^2\,k^2\,\tau_a}{\omega^2\,\varphi_a^2 - \Omega_a^2}\right) - \frac{\omega^2\,\varphi_a\,\Pi_a^2}{\omega^2 - c^2\,k^2}\,\sigma_b\,, \tag{5.4}$$

$$C_{ab} = (\omega^2\,\varphi_a^2 - 3\,c^2\,k^2\,\tau_a)\,\frac{\sigma_a}{\varphi_a}\,\delta_{ab} - \Pi_a^2\,\frac{\sigma_b}{\varphi_b}\,, \tag{5.5}$$

$$F_{ab} = \omega\,\Omega_a\,\sigma_a\,\delta_{ab}\left(1 + \frac{c^2\,k^2\,\tau_a}{\omega^2\,\varphi_a^2 - \Omega_a^2}\right) \tag{5.6}$$

with the ratio $\tau_a^{1/2}$ of the thermal velocities to the velocity of light given by

$$\tau_a = P_a/c^2\,\varrho_a\,. \tag{5.7}$$

We introduce the $N \times N$ matrices

$$\mathbf{A} \equiv (A_{ab})\,, \quad \mathbf{C} \equiv (C_{ab})\,, \quad \mathbf{F} \equiv (F_{ab}) \tag{5.8}$$

and three column matrices with N elements

$$\mathbf{W}_j \equiv (v_j^b) \quad (j = 1, 2, 3) \tag{5.9}$$

which allow us to recast the set (5.1—3) into

$$\begin{cases} \mathbf{A}\cdot\mathbf{W}_1 - i\,\mathbf{F}\cdot\mathbf{W}_2 = 0\,,\\ i\,\mathbf{F}\cdot\mathbf{W}_1 + \mathbf{A}\cdot\mathbf{W}_2 = 0\,,\\ \mathbf{C}\cdot\mathbf{W}_3 = 0\,. \end{cases}$$

The first two of these equations can be melted into one matricial equation

$$\mathbf{M}\cdot\mathbf{W}_2^1 = 0$$

with

$$\mathbf{M} \equiv \begin{pmatrix} \mathbf{A} & -i\,\mathbf{F}\\ i\,\mathbf{F} & \mathbf{A} \end{pmatrix}, \quad \mathbf{W}_2^1 \equiv \begin{pmatrix} \mathbf{W}_1\\ \mathbf{W}_2 \end{pmatrix}.$$

For a beam-plasma system with N constituent parts, $\mathbf{M}$ is a square matrix of rank $2N$ and $\mathbf{C}$ one of rank N.

6. Dispersion Relations

If we require at least one set of nonvanishing components for the first-order drift velocities, we must either put det $\mathbf{M}$ or det $\mathbf{C}$ equal to zero. In the former case, which is equivalent with

$$\det(\mathbf{A} \pm \mathbf{F}) = 0 \tag{6.1}$$

as was shown in (I.8.3), we find the dispersion relations for the transverse waves, whereas the longitudinal waves are given by the latter case:

$$\det\mathbf{C} = 0\,. \tag{6.2}$$

The discussion of (6.1) requires the computation of $A_{ab} \pm F_{ab}$ from (5.4) and (5.6):

$$A_{ab} \pm F_{ab} = \left(\omega\,\varphi_a \pm \Omega_a - \frac{c^2\,k^2\,\tau_a}{\omega\,\varphi_a \pm \Omega_a}\right)\omega\,\sigma_a\,\delta_{ab} - \frac{\omega\,\varphi_a\,\Pi_a^2}{\omega^2 - c^2\,k^2}\,\omega\,\sigma_b\,. \tag{6.3}$$

A comparison between the explicit forms of the elements of det $(\mathbf{A} \pm \mathbf{F})$ and of det $\mathbf{C}$, as given respectively by (6.3) and (5.5) learns us that both have a similar structure. Hence when set equal to zero, we can use for these determinants the development from (I.7.3) to (I.7.5).

For the *transverse waves* we have to put

$$X_a = \omega \varphi_a \pm \Omega_a - c^2 k^2 \tau_a / (\omega \varphi_a \pm \Omega_a), \qquad Y_a = \omega \varphi_a \Pi_a^2 / (\omega^2 - c^2 k^2), \qquad Z_a = \omega \sigma_a$$

to get that

$$1 = \sum_a \frac{\omega \varphi_a \Pi_a^2 / (\omega^2 - c^2 k^2)}{\omega \varphi_a \pm \Omega_a - c^2 k^2 \tau_a / (\omega \varphi_a \pm \Omega_a)}$$

or explicitly

$$R^2 = 1 - \sum_a \frac{\omega - k V_a}{\omega - k V_a \pm \Omega_a - c^2 k^2 \tau_a / (\omega - k V_a \pm \Omega_a)} \cdot \frac{\Pi_a^2}{\omega^2} - \sum_b \frac{\Pi_b^2}{\omega(\omega \pm \Omega_b - c^2 k^2 \tau_b / (\omega \pm \Omega_b))} \tag{6.4}$$

if

$$R = c k / \omega \tag{6.5}$$

is to be thought of as a refractive index of the beam-plasma system with respect to the vacuum. In (6.4) the summation over a is carried out for all components of our beam-plasma system which have nonzero drift velocities in the equilibrium state, the beam components, whereas the summation over b is for components at rest in the steady state, the proper plasma components. Hence (6.4) is applicable to any beam-plasma system.

The *dispersion relation for the longitudinal waves* is also obtained from (I.7.5) with

$$X_a = \omega^2 \varphi_a^2 - 3 c^2 k^2 \tau_a,$$
$$Y_a = \Pi_a^2,$$
$$Z_a = \sigma_a / \varphi_a$$

and hence

$$1 = \sum_a \Pi_a^2 / (\omega^2 \varphi_a^2 - 3 c^2 k^2 \tau_a). \tag{6.6}$$

Due to the low-temperature condition

$$c^2 k^2 \tau_a \ll \omega^2$$

the explicit expression of (6.6) is

$$1 = \sum_a \frac{\Pi_a^2}{(\omega - k V_a)^2} \left(1 + 3 \frac{c^2 k^2 \tau_a}{(\omega - k V_a)^2}\right) + \sum_b \frac{\Pi_b^2}{\omega^2} \left(1 + 3 \frac{c^2 k^2 \tau_b}{\omega^2}\right) \tag{6.7}$$

where the same summation convention is adopted as for the transverse waves. Hence (6.7) is also likeable to exhibit the same wide range of applications as (6.4).

7. Special Cases

One much studied special case is the cold beam-plasma system. Putting every τ_a and τ_b equal to zero in (6.4) and (6.7) yields respectively

$$R^2 = 1 - \sum_a \frac{\omega - k V_a}{\omega - k V_a \pm \Omega_a} \frac{\Pi_a^2}{\omega^2} - \sum_b \frac{\Pi_b^2}{\omega(\omega \pm \Omega_b)} \tag{7.1}$$

and

$$1 = \sum_a \Pi_a^2 / (\omega - k V_a)^2 + \sum_b \Pi_b^2 / \omega^2. \tag{7.2}$$

If the summation over b in (7.1) is restricted to the familiar electron and one ion species only, we recover the corresponding result of VLAARDINGERBROEK-WEIMER [3].

(7.2) On the other hand is the generalization for multiple particle beams in the presence of a multi-component plasma of the problem studied by DAWSON[4]. If we restrict ourselves in (7.2) to one kind of particles, in particular electrons, travelling in beams with different velocities, without any plasma at rest, we get his formula

$$1 = (q_e^2 / \varepsilon_0 m_e) \sum_a N_a / (\omega - k V_a)^2.$$

In the absence of an external magnetic induction (7.1) becomes indifferent for any change in the zero-order drift velocities V_a. That formula could hence be computed in the limit V_a equal to zero, e.g. from STIX [5].

[3] M. T. VLAARDINGERBROEK and K. R. U. WEIMER, Philips Res. Repts. **18**, 95 [1963].

[4] J. M. DAWSON, Phys. Rev. **118**, 381 [1960].

[5] T. H. STIX, The Theory of Plasma Waves, McGraw-Hill Book Co., New York 1962.

8. Conclusions

As an application of the low-temperature approximation, we were able to find the dispersion relations for linear waves in multicomponent beam-plasma systems, in the case of wave propagation and beam direction parallel to the external magnetic induction.

These formulas have a very wide range of applications, as they are not subject, to begin with, to the CHEW-GOLDBERGER-LOW hypothesis of strong external magnetic inductions. Amongst others, their applications include the now conventional multicomponent plasmas, charged particle beams and every physical situation between the two extremes cited. Some notable cold beams and plasmas results are included as special cases, but are by no means intended to be the only applications possible. The applications discussed in the last sections are meant as an illustration of the general treatment but not as an exhaustive procedure.

We wish to thank Professor dr. R. MERTENS for his stimulating interest in our work.

Über das unterschiedliche Ionisationsgleichgewicht wasserstoff- und alkali-ähnlicher Ionen in optisch dünnen Plasmen *

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(Z. Naturforschg. **22 a**, 1939—1952 [1967]; eingegangen am 14. Juli 1967)

There are many plasmas in which the populations of the various energy states of ions and electrons assume a steady state, but complete local thermal equilibrium is prevented because, on the one hand, radiation absorption is absent and, on the other, the electron density is too low.

Calculations for a plasma without radiation absorption show that: Where hydrogen-like ions are involved it is nearly in order to dispense with detailed calculations and described the ionization equilibrium with the assumption that the collisional ionization processes from the ground state can be equated with the radiative recombination processes to the ground state. For sufficiently low electron densities this leads to the Corona formula.

In the case of other ions, however, neglecting the excited states may result in serious errors. This is because these energy states — unlike those for hydrogen-like ions, which are relatively near the ionization limit — are distributed much more uniformly between the ground state and ionization limit. The implications of this behaviour are discussed with reference to alkali-like ions. A model for the term systems and the collision and radiation coefficients is used to derive the population densities and approximative ionization formulae. According to these the ratio of the densities of the lithium-like O VI ions and the next higher level of ions (O VII), for instance, may differ (for electron densities $n_e = 5 \times 10^{17}$ cm^{-3}) from the result of the Corona formula by a factor of 20.

Bei der spektroskopischen Plasmadiagnostik hat man es häufig mit Plasmen zu tun, deren Elektronendichten zu gering sind, um noch Anregungs- und Ionisationsverhältnisse zu gewährleisten, wie sie die Formeln des vollständigen thermischen Gleichgewichts (LTE) beschreiben. Dabei können aber ihrerseits die Partialdichten der beobachteten Ionenart so klein sein, daß man von jeder Art von Photoabsorptionsprozessen absehen kann. Das beides trifft häufig auf sogenannte Verunreinigungs-Ionen zu, die den Hochtemperaturplasmen, beabsichtigt oder nicht, zugefügt sind. In diesem Zusammenhang wird ein derartiges Plasma gewöhnlich als optisch dünn bezüglich dieser Ionen bezeichnet.

Die vorliegenden Rechnungen befassen sich besonders mit dem Fall, daß die Ionen, für die das Plasma als optisch dünn zu betrachten ist, der isoelektronischen Sequenz des Lithiums angehören. Dazu zählen u.a. C IV, N V und O VI, das sind Ionen, die in vielen Plasmen gegenwärtig sind und auch oft zu spektroskopischen Messungen herangezogen werden. Zwar umschließt das hier vorgeschlagene Modell alle alkaliähnlichen sowie auch die wasserstoffähnlichen Ionen, doch wird insbesondere das Ergebnis für lithiumähnliche Ionen benützt, um auf einen entscheidenden Unterschied gegenüber den wasserstoffähnlichen hinzuweisen.

* Auszug aus der von der Fakultät für Maschinenwesen und Elektrotechnik der Technischen Hochschule München genehmigten Dissertation über „Das unterschiedliche Ionisationsgleichgewicht wasserstoff- und alkaliähnlicher Ionen in optisch dünnen Plasmen“ des Dipl.-Phys. CARSTEN MAHN.

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